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ABSTRACT

A work-scheduling model for Corrections Officers at State Correctional Institutions is described. This is a real-life model that was developed to deal with a problem of unacceptably large expenditures for overtime work by state prison guards. The problem involves more than 200 constraints and more than 800 variables. It is felt the model can be described and understood without specialized knowledge in any particular field of study. Sections cover: 1) History of the Problem; 2) General Discussion of the Work-Scheduling Model; 3) Mathematical Description of the Model; and 4) Comparison of Results from the Model with Past Data from Two Prisons. The module also contains Concluding Remarks, References, Acknowledgements, and a Final Exam. (MP)

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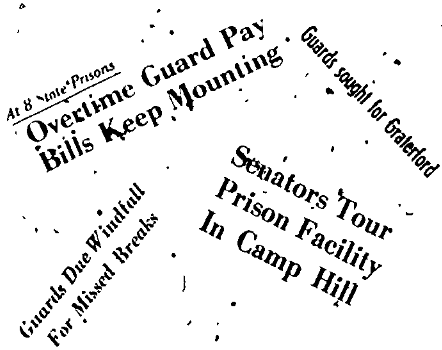
UMAP

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

MODULE 272

A Linear Programming Model for Scheduling Prison Guards

by James M. Maynard



Applications of Linear Programming to Operations Research

Birkhäuser Boston Inc.
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Cambridge, MA 02139

A LINEAR PROGRAMMING MODEL FOR

SCHEDULING PRISON GUARDS

by

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Intermodular Description Sheet: UMAP Unit 272

Title: A LINEAR-PROGRAMMING MODEL FOR SCHEDULING PRISON GUARDS

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Review Stage/Date: IV 4/30/80

Suggested Support Material: Access to a computer.

Prerequisite Skills:
1. Understand the basic form of the general linear programming model.

Output Skills:
1. Construct similar work-scheduling models.
2. Modify the model to fit different conditions.

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The Associated Press, "At 8 State Prisons Overtime Guard Pay Bills Keep Mounting," as appearing in The Patriot, Harrisburg, PA, August 16, 1975, p. 6.
The Associated Press, "Guards Sought for Graterford," as appearing in Evening News, Harrisburg, PA, December 16, 1974, Section 2, p. 1.
The Patriot Wire Services, "Guards Due Windfall for Missed Breaks," The Patriot, Harrisburg, PA, August 5, 1974, p. 12.
Brooks, Merry, "Senators Tour Prison Facility in Camp Hill," The Patriot, Harrisburg, PA, May 2, 1974, p. 59.

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1. HISTORY OF THE PROBLEM

1.1 Introduction

This paper describes a work-scheduling model for Corrections Officers (primarily prison guards) at State Correctional Institutions (state prisons). This is a real-life model that was developed to deal with a very real problem: unacceptably large expenditures for overtime work by state prison guards (these expenditures are described in Section 1.2). It is not a typical "textbook example." In particular, the problem is larger than any textbook examples in the author's experience; the model involves more than 200 constraints and more than 800 variables, as described in Section 3.5. (However, this is not an especially large "real world" linear programming problem.) The notation necessary to describe a problem of this magnitude is unavoidably cumbersome, as will be seen in Section 3.2. On the other hand, the model can be described and understood without specialized knowledge in any particular field of study.

Difficulties (unrelated to the model per se) were encountered in the collection and interpretation of data required to evaluate the performance of the model; these difficulties (typical of the "real world"), and the procedures used to overcome them, are described in Section 4.

Because prison guards come only in whole units (no fractional parts), a special technique was used to obtain an integer-valued solution to the model; this is described, along with an intuitive justification, in Section 3.6. At present, this intuitive justification, and the complete success achieved in obtaining satisfactory integer-valued solutions in the test cases that have been run (as described in Section 4), are the only justification for this special technique. There is no formal proof given here that the procedure will always yield an interger-valued solution, and no such proof is presently known to the author. It is well-known that under commonly-satisfied conditions, the so-called "transportation problem" will always possess an integer-valued optimal solution; possibly a proof of this result (e.g., as given in Section 9-4 of [1]) could be adapted to fit the present model. Or, perhaps there are situations for which the special procedure used here will not yield an interger-valued solution. Presently, this is still an open question. Other, more urgent problems have precluded further work on this question. However, the procedure described here in Section 3.6 has provided satisfactory integer-valued solutions in the real-life cases that have been run, and so, in the practical sense, has "solved" the problem. In real-life applications of mathematics, this is often what matters--"the proof of the pudding is in the eating."

Numerical results from applying the model for two different state prisons are given in Section 4, along with some interpretive comments.

The exercises and final exam included here are designed to aid understanding of this particular model, so as to illustrate the detailed consideration necessary for modeling real-life situations.

Solution of such a large model would not be possible without large-scale computing facilities. Because of the size of the model and the variation in input/output formats for various "canned" linear programming computer software packages, no computer runs are included here. If the reader wants experience in running such a model on a locally-available computer, the small-scale version of the model given in the final exam could be used, utilizing work-force requirements chosen by the reader as input data.

The model described here was developed after policies and procedures for work scheduling at various state prisons had been judged to need improvement. This judgment was based largely on the following two considerations:

(1) work schedules varied from one prison to another, causing undesirable variations from prison to prison in such matters as the number of days off for guards.

(2) work-force requirements for prison guards (i.e., the required numbers of prison guards on duty) at each prison were being met by scheduling large amounts of overtime work which could cause fatigue, inefficiency, tension, etc., as well as increased labor costs.

1.2 Overtime Costs

The cost of overtime work was a serious problem. Some indication of the size of overtime cost can be obtained from Tables 1 and 2. Table 1 shows the number of prison guards on duty at one state prison, designated Prison H, during the week ending September 23, 1973. In Table 1, for each eight-hour period of each day in the week, four numbers are given without parentheses. (Temporarily ignore the numbers in parentheses; they will be discussed in Section 4.1). In order from top to bottom, the four numbers without parentheses are:

(1) the number of prison guards working that period as part of their regular forty-hour-per-week work

TABLE 1

Data and Results from Prison H for the Week Ending September 23, 1973

DAY	MORNING		AFTERNOON		NIGHT	
MONDAY	40	(40)	41	(41)	20	(20)
	0	(0)	0	(0)	0	(0)
	0	(0)	0	(0)	0	(0)
	40	(40)	41	(41)	20	(20)
TUESDAY	44	(45)	41	(41)	19	(19)
	1	(0)	0	(0)	0	(0)
	0	(0)	0	(0)	0	(0)
	45	(45)	41	(41)	19	(19)
WEDNESDAY	44	(48)	37	(40)	18	(13)
	0	(1)	0	(0)	0	(5)
	5	(0)	3	(0)	0	(0)
	49	(49)	40	(40)	18	(18)
THURSDAY	42	(42)	39	(40)	17	(14)
	1	(1)	0	(0)	0	(3)
	0	(0)	0	(0)	0	(0)
	43	(43)	39	(40)	17	(17)
FRIDAY	44	(45)	39	(40)	16	(16)
	1	(0)	1	(0)	0	(0)
	0	(0)	0	(0)	0	(0)
	45	(45)	40	(40)	16	(16)
SATURDAY	36	(37)	38	(38)	16	(16)
	1	(0)	0	(0)	0	(0)
	0	(0)	0	(0)	0	(0)
	37	(37)	38	(38)	16	(16)
SUNDAY	33	(33)	40	(40)	17	(17)
	0	(0)	0	(0)	0	(0)
	0	(0)	0	(0)	0	(0)
	33	(33)	40	(40)	17	(17)

schedule, at regular rate of pay (e.g., there were 44 of these on Tuesday Morning);

(2) the number of guards working that period on overtime at one-and-a-half (1.5) times their regular rate of pay (e.g., there was one of these on Tuesday Morning);

(3) the number of guards working that period on overtime at two (2) times their regular rate of pay (e.g., there were three of these on Wednesday Afternoon);

(4) the total number of guards on duty during that period, which is just the sum of the previous three numbers (e.g., there were 49 guards working the Wednesday Morning period).

The reason that some guards working overtime were paid "time-and-a-half" while others were paid "double-time" will be explained in assumption (10) of Section 2.1. Table 2 gives the same kind of data for another prison, designated prison G, for the week ending September 30, 1973.

Table 1 indicates that the equivalent of five periods was worked at time-and-a-half during that week at Prison H: one each on Tuesday Morning, Thursday Morning, Friday Morning, Friday Afternoon, and Saturday Morning. Also, the equivalent of eight periods was worked at double-time: five on Wednesday Morning and three on Wednesday Afternoon. For convenience, let us assume that all prison guards were paid \$3.00 per hour as regular rate (actually, the minimum pay for a prison guard is higher than this). Then since each period is eight hours, Table 1 indicates a total cost for overtime work of

$$(1.5)(\$24)(5) + (2)(\$24)(8) = \$564$$

for that week at Prison H.

Exercise 1.2 Overtime Cost at Prison G:

Again assuming that all prison guards were paid \$3 per hour as standard rate, verify that Table 2 indicates a total cost for overtime work of \$23,928 at Prison G during the week ending September 30, 1973. (Prison G is a considerably larger prison that Prison H.)

These large amounts of overtime pay were also being noticed by the news media, as shown by the newspaper clippings reproduced here on pages 6 and 7. (These clippings are reproduced with the kind permission of the Associated Press and the Harrisburg Patriot and Evening News.).

TABLE 2

Data and Results from Prison G for the Week Ending December 30, 1973

DAY	SHIFT					
	MORNING*		AFTERNOON		NIGHT	
MONDAY	94	(117)	70	(131)	38	(74)
	19	(0)	61	(0)	40	(4)
	3	(0)	0	(0)	0	(0)
	116	(117)	131	(131)	78	(78)
TUESDAY	94	(126)	70	(137)	36	(74)
	17	(0)	62	(9)	38	(0)
	15	(0)	14	(0)	0	(0)
	126	(126)	146	(146)	74	(74)
WEDNESDAY	97	(116)	69	(137)	36	(74)
	19	(0)	68	(0)	27	(1)
	0	(0)	0	(0)	12	(0)
	116	(116)	137	(137)	75	(75)
THURSDAY	94	(128)	63	(98)	37	(74)
	41	(21)	24	(2)	34	(7)
	14	(0)	13	(0)	10	(0)
	149	(149)	100	(100)	81	(81)
FRIDAY	74	(97)	45	(89)	37	(39)
	20	(0)	16	(0)	2	(0)
	2	(0)	0	(0)	0	(0)
	96	(97)	61	(89)	39	(39)
SATURDAY	57	(43)	37	(45)	26	(0)
	15	(33)	14	(6)	3	(29)
	4	(0)	0	(0)	0	(0)
	76	(76)	51	(51)	29	(29)
SUNDAY	53	(63)	36	(48)	25	(35)
	7	(0)	12	(0)	3	(0)
	3	(0)	0	(0)	2	(0)
	63	(63)	48	(48)	30	(35)

Senators Tour Prison Facility In Camp Hill

By MERRY BROOKS
Staff Writer

A fact-finding tour by members of the State Senate Prison Inquiry Committee yesterday at the State Correctional Institution at Camp Hill seemed more like a whirlwind campaign swing with senators shaking hands and eliciting opinions from prisoners and guards

But in fact, the information sought by four state senators on the sixth tour of the eight state prisons yielded information that may assist the special committee in drafting prison-related legislation.

Sen. Freeman Hankins, D-Philadelphia, committee chairman; Sen. Martin Murray, D-Luzerne; Sen. Herbert Arlene, D-Philadelphia, and Sen. James E. Ross, D-Beaver-Washington, accompanied by a herd of reporters, breezed through the prison in Lower Allen Twp in a three-hour VIP tour.

The senators engaged Ernest Patton, prison superintendent, in a give-and-take roundtable discussion before the tour began. They obtained the following information.

—The prison paid $353,000 in overtime to guards last year and expects to pay $361,000 in overtime this year. The prison needs an additional 67 guards to reduce the amount of overtime pay.

Courtesy of the Harrisburg Patriot.

Guards sought for Graterford

By The Associated Press

The head of the state Corrections Bureau says Gov Shapp and the Legislature may be asked to provide from 80 to 100 guards at the Graterford State Prison

The increase would raise to 400 the number of guards at the Montgomery County prison.

Corrections Commissioner Stewart Werner estimated the added guards would cost $500,000 annually.

The extra men could cut down on overtime payments to the current guards, now running about $20,000 a month.

★ ★ ★

Graterford, the largest of the state's eight correctional institutions, has about 1,800 inmates, about 200 below capacity.

Courtesy of the Associated Press.

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At 8 State Prisons

Overtime Guard Pay Bills Keep Mounting

By The Associated Press

The Bureau of Corrections is still paying heavy overtime to keep guards on duty at the eight state prisons. Some guards are doubling their salaries through extra work.

A bureau spokesman said yesterday that in the year ended June 30 the agency paid out nearly $4 million in overtime, a boost of $750,000 over the previous year.

The bureau already had been strongly criticized for excessive overtime payments. Legislators and other officials think the state could save money by hiring more guards at regular salaries and reducing overtime payments at time and a half and double time.

Auditor Gen. Robert P. Casey, one of the critics, zeroed in yesterday on overtime at the state prison in Dallas, Luzerne County. More than $430,874 was paid during the fiscal year ended in June 1974.

Casey said 12 guards received between $10,144 and $6,707 in overtime. Eleven of the 12 guards had base salaries of $11,731. One guard has a $12,875 base salary.

"The new commissioner — William Robinson — is very aware of the problem and that, along with every other program, is being looked at very carefully ...," the correction bureau spokesman said. "He does want to cut down on the overtime."

He said former Commissioner Stewart Werner had a hiring freeze in effect because of the tight budget policy adopted by the Shapp administration.

But under Robinson, who assumed the post last month, the freeze has been lifted and 35 guard vacancies around the state are being filled, the spokesman said.

However, the overtime problem will linger.

Glen R. Jeffes, superintendent at Dallas, said vacancies alone don't govern how much overtime will be needed. Vacations and the fact that authorized staff levels are inadequate also are factors, he said.

"I have requested additional officer positions the last two years. I received no new positions ..."

"Without additional officer positions I see very little impact on the reduction of overtime," he said.

Courtesy of the Associated Press.

Guards Due Windfall For Missed Breaks

From The Patriot Wire Services

About 1,700 state prison guards will be reimbursed for perhaps $1,000 each for missed coffee breaks, it was learned yesterday.

The windfall comes as a result of an arbitrator's decision earlier this month on grievances filed at eight penal institutions across the state. It may cost the commonwealth as much as $1.7 million.

Under the terms of their contract with the State Bureau of Corrections, the guards are allowed a 15-minute break every four hours.

But because of critical manpower shortages at the state's prisons, the men have not been able to take the breaks since before July, 1973.

Robert Saylor, executive director for the Bureau of Corrections, refused to comment on published reports of the reimbursement.

The guard answering the phone at the bureau's headquarters here said he had heard of the decision, but added, "We should be getting $2,000."

The arbitrator's decision was handed down on July 12, according to published reports, but the bureau made no announcement of it.

According to Jack Walsh, president of guards Local 2500 at Western State Penitentiary, the payment will be made to the guards sometime this month.

Courtesy of the Harrisburg Patriot.

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1.3 Beginning the Study for a Better Scheduling Procedure

For the reasons indicated above, a study was initiated to determine if prison guards could be utilized more efficiently than was occurring with the procedures then in use by the various state prisons (as discussed in Section I.1, the work-scheduling patterns then in use varied from prison to prison). At the beginning of this project, the following objectives were considered:

(1) provide a uniform work-scheduling pattern for all state prisons;

(2) eliminate excessive overtime work by the prison guards;

(3) indicate the optimal number of prison guards to be employed at each prison so as to minimize total labor costs while meeting specified work-force requirements;

(4) provide a uniform pattern of shift rotation for the guards at each prison.

It was soon decided that the selection of individual prison guards to fill the various work assignments available should be left in the hands of management to provide maximum flexibility in scheduling prison guards with special abilities, work preferences, etc., as well as to keep the scope of the present project within reasonable bounds (which could not be done if account had to be taken of each individual prison guard). For similar reasons, the pattern of shift rotation for the prison guards at each prison (objective (4) above) was omitted from further consideration. So it was decided to try to develop a work-scheduling model which would meet the first three of the four objectives listed above. The model is intended to provide the basis for routine scheduling of the guards. Unexpected absenteeism caused by illness, etc., will always have to be dealt with separately as it occurs.

2. GENERAL DISCUSSION OF THE WORK-SCHEDULING MODEL

2.1 Assumptions

Each prison is considered as a completely separate entity. Then the model, as applied to each prison individually, is based on the assumptions listed below. (In the following discussion, the word <u>shift</u> refers to one of the three divisions of a day: Morning, Afternoon, or Night. The word <u>period</u> refers to a single eight-hour length of time which is determined by specifying both a day of the week and a shift of the day, e.g., the Wednesday Afternoon period.)

(1) Each twenty-four-hour day is divided into three eight-hour shifts: Morning, Afternoon, and Night (see Tables 1 and 2).

(2) Work-force requirements (i.e., the numbers of guards required to be on duty) are known for each period of each day for a full seven-day week. Furthermore, these requirements do not change from week to week. Also, the total number of prison guards available for work is constant (does not change during a week or from week to week). In other words, the model developed here is a static model for the scheduling of guards on a weekly basis; it is not a dynamic model. (In this respect, the model is similar to the one discussed by J.A. Parsons [2].) See Section 2.3 for further discussion of this assumption and a method to introduce limited dynamic qualities into the model.

(3) As a standard weekly tour of duty, each prison guard is assigned to a forty-hour standard-rate work schedule consisting of one eight-hour period per day for five consecutive days, staying on the same shift for all five days; he/she then has two consecutive days off before beginning this same pattern again. For example, some guards work the Morning shift Monday through Friday; other guards work the the Night shift Wednesday through Sunday, etc. There are twenty-one of these forty-hour standard-rate work schedules: one each beginning with Monday Morning, Monday Afternoon, Monday Night, Tuesday Morning, etc., to Sunday Night (see Section 3.2). Since this is a static model, these forty-hour standard-rate work schedules are assumed to repeat without change week after week (but see Section 2.3).

(4) Work-force requirements not met by these standard-rate work schedules are filled by overtime assignments; each overtime assignment is for one eight-hour period.

(5) A prison guard is not eligible for overtime work on any of the thirteen periods which occur during his/her standard forty-hour work week; i.e., during the time commencing with the first standard work period of his/her five-day work schedule. For example, a guard who works the Morning shift Monday through Friday as his/her standard forty-hour work week would not be eligible for overtime

work anytime between Monday Morning period and Friday Morning period, inclusive.

(6) A guard is eligible for overtime work on any of the eight periods which occur during his/her two days off, commencing with the period immediately following his/her fifth standard work period and terminating with the period immediately preceding the first standard work period of his/her next five-day work schedule. For example, a guard who works the Morning shift Monday through Friday as his/her standard forty-hour work week would be eligible for overtime work on any of the eight periods between Friday Afternoon and Sunday Night, inclusive.

(7) However, no guard may work more than two overtime periods during his/her two days off.

(8) As implied by the above assumptions, a guard may work two or even three consecutive eight-hour periods, depending on what combinaton of standard and overtime periods he/she is assigned. See Section 2.4 for discussion of modifying the assumption in the model.

(9) All prison guards are assumed to be paid at the same (unspecified) standard rate for an eight-hour period, with no account being taken of differences in pay because of differences in seniority, skills, etc. (Otherwise, minimization of overtime costs would require that overtime work for each week always be assigned to guards receiving the lowest rate of pay.)

(10) If a guard does overtime work durig his/her two days off, then for the first overtime period during the two days off he/she is paid one-and-a-half (1.5) times his/her standard rate of pay. If he/she works a second over-time period during the same two days off, then for this second overtime period he/she is paid two (2) times his/her standard rate of pay. (The numbers 1.5 and 2 specified above can be changed easily in the model.)

2.2 Input Requirements of the Model and Resulting Output

For a given prison, the model needs the work-force requirements for each period of each day of the week (twenty-one requirements in all) as input data. The model will then determine an "optimal" size of the work force (total number of guards), an "optimal" number of guards to be assigned to each of the twenty-one forty-hour standard-rate work schedules, and an "optimal" number of guards to be assigned to each period of each day on an overtime basis, so as to "minimize" total labor costs while meeting the conditions stipulated in the assumptions of Section 2.1 and at least satisfying the specified work-force requirements. (In this paper, the words "optimal" and "minimize," when in quotes, will mean <u>nearly optimal</u> and <u>nearly minimal</u>, repectively, because of the procedure used to obtain an integer-valued solution; see Section 3.6.) The model will indicate which of the one-and-a-half-rate overtime assignments are to be filled from each forty-hour standard-rate work schedule, and which of the double-rate overtime assignments are to be filled from each one-and-a-half-rate overtime assignment so as to satisfy assumptions (4), (5), (6), and (7) in Section 2.1. Finally, the model will indicate the total labor cost resulting from this overall work pattern; the cost is given as a multiplier of the cost for a prison guard to work one standard-rate eight-hour period. For example, if guards are paid $3 an hour as standard rate (a figure used for illustrative purposes only), then the total labor cost given by the model must be multiplied by $24 to express total labor cost in dollars. Recall from Section 1.3 that the selection of individual prison guards to fill the various work assignments indicated by the model is left in the hands of management, to provide maximum flexibility in scheduling guards with special abilities, work preferences, etc.

2.3 Modifying the Model for a Fixed Work Force

Very slight modification of the model will enable a user to specify the size of the work force (total number of guards) as input data along with the twenty-one work-force requirements. In this case, for a fixed total work force, the cost of standard-rate work is fixed (assuming, as in assumption (3) of Section 2.1, that each guard works at least forty hours per week). So, the objective of minimizing total labor costs now becomes equivalent to minimizing overtime costs only. The model will now determine an "optimal" number of guards to be assigned to each of the twenty-one forty-hour standard-rate work schedules, and an "optimal" number of guards to be assigned to each period of each day on an overtime basis, so as to "minimize" overtime costs while meeting the conditions stipulated in the assumptions of Section 2.1 and at least satisfying the specified work-force requirements. Other features of the model output are similar to the variable-work-force case discussed in Section 2.2, except that overtime cost is given instead of total manpower cost. Further discussion of this modification is given in Section 4.1.

This flexibility makes it possible to use the model to determine a new set of work-force assignments using a previously-determined total work force, in response to a perhaps seasonally-adjusted pattern of work-force requirements, with relatively minor "end effects" necessary to smooth the transition from one set of work-force assignments to the next. Since past policies, union contracts, availability of trained manpower, etc., all restrict the total work force from week to week, this limited flexibility may provide sufficient dynamic character to the model.

2.4 Modifying the Model to Prevent Consecutive Work Periods

When a prison guard works two or three consecutive eight-hour periods as indicated in assumption (8) of Section 2.1, then fatigue, inefficiency, tension, etc., will surely result. Such conditions can have serious consequences in a situation requiring delicate interpersonal relationships, such as occur in a prison. The work-scheduling model described here can be modified to impose restrictions against any guard working three consecutive eight-hour periods, or against any guard working even two consecutive eight-hour periods. More generally, the model can be modified to impose any desired number of "rest" periods between those periods for which a guard is considered eligible for overtime work. This eligibility depends, of course, on which forty-hour standard-rate work schedule the guard is assigned. Of course, the number of rest periods must be consistent with assumptions (4), (5), (6), and (7) in Section 2.1. This modification merely requires the deletion of certain terms from the mathematical expressions representing assumptions (6) and (7), and the removal of the corresponding terms from the computer-card decks with which the model is implemented. The modification is tedious but straightforward. See Section 3.4 for further details.

2.5 Overstaffing

As indicated in Sections 2.2 and 2.3, the solution provided by the model will at least satisfy the specified work-force requirements. In other words, the total number of prison guards (standard-rate and overtime) assigned to any period of any day will at least equal the corresponding work-force requirement for that period. Because standard-rate work schedules are assigned in forty-hour "blocks," it is possible that the number of standard-rate guards assigned to a particular period of some day may exceed the corresponding work-fore requirement for that period. Prior knowledge of this situation will enable management to efficiently schedule any anticipated extra work to be done during such overstaffed periods. Of course, when a period is already overstaffed by standard-rate guards, the model will not assign any overtime for that period. Also, this overstaffed condition should not be misinterpreted; it still represents a "minimum-cost" solution to the work-scheduling problem described by the model, using the twenty-one specified work-force requirements. Examples of such overstaffed periods will be given in Section 4.

3. MATHEMATICAL DESCRIPTION OF THE MODEL

3.1 Preliminary Comments

The mathematical model developed in this section for minimizing total labor costs in accordance with the previous discussion is an integer linear programming model, i.e., a linear programming model in which the decision variables must be integer valued. Very good discussions of the theory and application of linear programming in modeling real-world problems are given in several books, including [3], [4]. Both of these books discuss the additional complications that can arise when the decision variables must be integer valued. We will return to this consideration for the present model in Section 3.6.

3.2 Notation

As a regular standard-rate weekly tour of duty, each prison guard is assigned to work eight hours per day for five consecutive days, staying on the same shift for all five days, and then to have two consecutive days off. The seven possible five-day work schedules are indicated below, with a corresponding value for an index k:

Work Schedule							k
M	T	W	Th	F			1
	T	W	Th	F	S		2
		W	Th	F	S	Su	3
M			Th	F	S	Su	4
M	T			F	S	Su	5
M	T	W			S	Su	6
M	T	W	Th			Su	7

The following notation will be used:

R_{ij} = required number of prison guards (work-force requirement) for day i and shift j.

N = total work force (number of prison guards).

x_{kj} = number of prison guards to be regularly assigned to work schedule k, shift j, at standard rate.

y_{ij} = number of prison guards to be assigned to day i, shift j, at one-and-a-half-rate overtime.

z_{ij} = number of prison guards to be assigned to day i, shift j, at double-rate overtime.

$u_{ij;pq}$ = number of prison guards chosen from x_{pq} to work one-and-a-half-rate overtime on day i, shift j.

$w_{ij(mn;pq)}$ = number of prison guards chosen from $u_{mn;pq}$ to work at double-rate overtime on day i, shift j
= number of prison guards chosen from x_{pq} to work at one-and-a-half-rate overtime on day m, shift n, and at double-rate overtime on day i, shift j.

Table 3 indicates the work-force assignments using the x_{kj}, y_{ij}, z_{ij} notation.

As examples of the notation, we have

R_{23} = required number of prison guards for Tuesday Night.

x_{42} = number of prison guards to be regularly assigned to Work Schedule 4 (Th, F, S, Su, M), Afternon shift, at standard rate.

y_{23} = number of prison guards to be assigned to Tuesday Night at one-and-a-half-rate overtime.

z_{32} = number of prison guards to be assigned to Wednesday Afternoon at double-rate overtime.

$u_{23;42}$ = number of prison guards chosen from x_{42} to work one-and-a-half-rate overtime on Tuesday Night.

$w_{32(23;42)}$ = number of prison guards chosen from $u_{23;42}$ to work double-rate overtime on Wednesday Afternoon.

Exercise 3.2 Notation:

(a) Write out the definitions of R_{62}, x_{53}, y_{31}, z_{72}, $u_{32;53}$, and $w_{51(32;53)}$.

(b) Is the symbol $u_{23;62}$ valid for this model? Explain.

(c) How many consecutive eight-hour shifts are worked by the group of prison guards represented by the symbol $w_{72(63;21)}$? $w_{42(41;53)}$? $w_{73(72;11)}$? $w_{61(52;62)}$?

(d) Without looking at Table 3, write out the expression representing the total number of prison guards to be working Tuesday Afternoon.

TABLE 3

Number of Prison Guards to be Assigned to Each Period of Each Day

DAY	SHIFT		
	Morning (a.m.) 6 a.m. - 2 p.m. $j = 1$	Afternoon (p.m.) 2 p.m. - 10 p.m. $j = 2$	Night (nt.) 10 p.m. - 6 a.m. $j = 3$
MONDAY $i = 1$	$x_{11}+x_{41}+x_{51}+x_{61}+x_{71}$ y_{11} z_{11}	$x_{12}+x_{42}+x_{52}+x_{62}+x_{72}$ y_{12} z_{12}	$x_{13}+x_{43}+x_{53}+x_{63}+x_{73}$ y_{13} z_{13}
TUESDAY $i = 2$	$x_{11}+x_{21}+x_{51}+x_{61}+x_{71}$ y_{21} z_{21}	$x_{12}+x_{22}+x_{52}+x_{62}+x_{72}$ y_{22} z_{22}	$x_{13}+x_{23}+x_{53}+x_{63}+x_{73}$ y_{23} z_{23}
WEDNESDAY $i = 2$	$x_{11}+x_{21}+x_{31}+x_{61}+x_{71}$ y_{31} z_{31}	$x_{12}+x_{22}+x_{32}+x_{62}+x_{72}$ y_{32} z_{32}	$x_{13}+x_{23}+x_{33}+x_{63}+x_{73}$ y_{33} z_{33}
THURSDAY $i = 4$	$x_{11}+x_{21}+x_{31}+x_{41}+x_{71}$ y_{41} z_{41}	$x_{12}+x_{22}+x_{32}+x_{42}+x_{72}$ y_{42} z_{42}	$x_{12}+x_{23}+x_{33}+x_{43}+x_{73}$ y_{43} z_{43}
FRIDAY $i = 5$	$x_{11}+x_{21}+x_{31}+x_{41}+x_{51}$ y_{51} z_{51}	$x_{12}+x_{22}+x_{32}+x_{42}+x_{52}$ y_{52} z_{52}	$x_{13}+x_{23}+x_{33}+x_{43}+x_{53}$ y_{53} z_{53}
SATURDAY $i = 6$	$x_{21}+x_{31}+x_{41}+x_{51}+x_{61}$ y_{61} z_{61}	$x_{22}+x_{32}+x_{42}+x_{52}+x_{62}$ y_{62} z_{62}	$x_{23}+x_{33}+x_{43}+x_{53}+x_{63}$ y_{63} z_{63}
SUNDAY $i = 7$	$x_{31}+x_{41}+x_{51}+x_{61}+x_{71}$ y_{71} z_{71}	$x_{32}+x_{42}+x_{52}+x_{62}+x_{72}$ y_{72} z_{72}	$x_{33}+x_{43}+x_{53}+x_{63}+x_{73}$ y_{73} z_{73}

3.3 The Objective Function: Total Labor Cost

Total labor cost includes overtime cost as well as standard-rate cost and fringe benefits. Now, fringe benefits are approximately equal to 20% of a prison guard's standard-rate pay, and are a cost in addition to his/her standard-rate pay. So, the total standard-rate pay for five eight-hour periods and corresponding fringe benefits is approximately equal to the cost of six standard-rate eight-hour periods (20% of five is one, and one plus five is six). So the total labor cost for all guards was taken as six times the number of guards on the work force, plus the total cost of all overtime assignments; this overtime cost was again formulated as a multiplier of the standard-rate cost for one prison guard to work an eight-hour period.

The objective function of our integer linear programming problem for the minimization of total labor cost is then

$$\text{Minimize} \quad Z = 6N + \sum_{i=1}^{7} \sum_{j=1}^{3} (1.5\, y_{ij} + 2z_{ij}).$$

If prison guards are paid $3 per hour, then this value of Z must be multiplied by $24 to give actual labor cost in dollars.

Exercise 3.3 Labor Cost

If fringe benefits of 20% were paid for overtime work as well as for standard-rate work, how then should the objective function representing total labor cost be written?

3.4 Constraints

The constraints for our model are

$$\sum_{k=1}^{7} \sum_{j=1}^{3} x_{kj} = N$$

$$\sum_{k} x_{kj} + y_{ij} + z_{ij} \geq R_{ij}$$

$$y_{ij} = \sum_{p,q} u_{ij;pq}$$

$$\sum_{i,j} u_{ij;pq} \leq x_{pq}$$

$$z_{ij} = \sum_{m,n} \sum_{p,q} w_{ij(mn;pq)}$$

$$\sum_{i,j} w_{ij(mn;pq)} \leq u_{mn;pq}$$

where the summations are taken over appropriate values of the indices consistent with the assumptions given in Section 2.1, and where all variables $\{x_{jk}\}$, $\{y_{ij}\}$, $\{z_{ij}\}$, $\{u_{ij;pq}\}$, $\{w_{ij(mn;pq)}\}$ are nonnegative integers. Note that the $\{y_{ij}\}$ and $\{z_{ij}\}$ are convenient notational and conceptual stand-ins for various sums of certain of the $\{u_{ij;pq}\}$ and $\{w_{ij(mn;pq)}\}$, but are superfluous to the mathematical description of the problem; the necessary variables for the mathematical description are the $\{x_{kj}\}$, $\{u_{ij;pq}\}$, and $\{w_{ij(mn;pq)}\}$.

For illustration, some typical examples of these constraints are given below. The underlining illustrates the modifications for preventing consecutive work periods mentioned in Section 2.4, as follows: triple underlining indicates a term representing three consecutive work periods, double underlining indicates a term representing two consecutive work periods, single underlining indicates a term representing one consecututve work period (i.e., at least one rest period between work periods), and no underlining indicates a term representing at least two rest periods between work periods. So, if prison guards are required to have at least two rest periods between work periods, then delete all the underlined entries in the following constraints. If guards are only required to have at least one rest period between work periods (i.e., are not permitted to work two consecutive eight-hour periods), then retain the singly-underlined entries (as well as the nonunderlined entries, of course) but delete the doubly-underlined and triply-underlined entries. If guards are permitted to work two consecutive periods, but are not permitted to work three consecutive periods, then retain the singly- and doubly-underlined entries, but delete the triply-underlined entries. Finally, if guards are permitted to work three consecutive periods, then retain all the entries.

$$x_{33} + x_{43} + x_{53} + x_{63} + x_{73} + y_{73} + z_{73} \geq R_{73};$$

$$y_{73} = \underline{\underline{u_{73;11}}} + \underline{u_{73;12}} + u_{73;13} + u_{73;21} + u_{73;22} + u_{73;23} + \underline{u_{73;31}} + \underline{\underline{u_{73;32}}};$$

$$\underline{\underline{u_{73;11}}} + \underline{u_{72;11}} + u_{71;11} + u_{63;11} + u_{62;11} + u_{61;11} + \underline{u_{53;11}} + \underline{\underline{u_{52;11}}} \leq x_{11};$$

$$
\begin{aligned}
z_{73} = {} & w_{73(72;11)} + w_{73(72;12)} + w_{73(72;13)} \\
& + w_{73(72;21)} + w_{73(72;22)} \\
& + w_{73(72;23)} + w_{73(72;31)} \\
& + w_{73(71;11)} + w_{73(71;12)} + w_{73(71;13)} \\
& + w_{73(71;21)} + w_{73(71;22)} \\
& + w_{73(71;23)} \\
& + w_{73(63;11)} + w_{73(63;12)} + w_{73(63;13)} \\
& + w_{73(63;21)} + w_{73(63;22)} \\
& + w_{73(62;11)} + w_{73(62;12)} + w_{73(62;13)} \\
& + w_{73(62;21)} \\
& + w_{73(61;11)} + w_{73(61;12)} + w_{73(61;13)} \\
& + w_{73(53;11)} + w_{73(53;12)} \\
& + w_{73(52;11)};
\end{aligned}
$$

$$
\begin{aligned}
& w_{73(52;11)} + w_{72(52;11)} + w_{71(52;11)} + w_{63(52;11)} \\
& \quad + w_{62(52;11)} + w_{61(52;11)} + w_{53(52;11)} \\
& \quad \le u_{52;11};
\end{aligned}
$$

$$
\begin{aligned}
& w_{73(53;11)} + w_{72(53;11)} + w_{71(53;11)} + w_{63(53;11)} \\
& \quad + w_{62(53;11)} + w_{61(53;11)} \le u_{53;11};
\end{aligned}
$$

$$
\begin{aligned}
& w_{73(61;11)} + w_{72(61;11)} + w_{71(61;11)} + w_{63(61;11)} \\
& \quad + w_{62(61;11)} \le u_{61;11};
\end{aligned}
$$

$$
\begin{aligned}
& w_{73(62;11)} + w_{72(62;11)} + w_{71(62;11)} + w_{63(62;11)} \\
& \quad \le u_{62;11};
\end{aligned}
$$

$$w_{73(63;11)} + w_{72(63;11)} + w_{71(63;11)} \le u_{63;11}.$$

$$w_{73(71;11)} + w_{72(71;11)} \le u_{71;11};$$

$$w_{73(72;11)} \le u_{72;11}.$$

Exercise 3.4 Constraints:

By analogy with the examples given above, complete the following constraints for our model, including the underlining as just discussed:

$x_{??} + \ldots \geq R_{52}$;	$y_{52} = u_{??;??} + \ldots$;
$u_{??;??} + \ldots \leq x_{52}$;	$z_{52} = w_{??(??;??)} + \ldots$;
$w_{??(??;??)} + \ldots \leq u_{23;52}$;	$w_{??(??;??)} + \ldots \leq u_{31;52}$;
$w_{??(??;??)} + \ldots \leq u_{32;52}$;	$w_{??(??;??)} + \ldots \leq u_{33;52}$;
$w_{??(??;??)} + \ldots \leq u_{41;52}$;	$w_{??(??;??)} + \ldots \leq u_{42;52}$;
$w_{??(??;??)} + \ldots \leq u_{43;52}$.	

3.5 Summary of the Mathematical Model

To summarize the mathematical model, it is convenient to imagine that each constraint is rewritten with all variables (N and the x's, y's, z's, u's, and w's) appearing on the left of the algebraic sign (=, $\geq$, or $\leq$) and only constants (the R_{ij}'s or 0) appearing on the right side of the sign. For example the "Total work force" constraint would be rewritten as

$$x_{11} + x_{21} + \ldots + x_{12} + x_{22} + \ldots + x_{73} - N = 0$$

with all 21 x's included. Then the model can be summarized as in Table 4. The matrix referred to in Table 4 consists of 821 columns (one for each variable and one for the right-hand-side terms of the rewritten constraints) together with 233 rows (one for the objective function and one for each constraint), giving a total of 191,293 cells.

A cell is the intersection of a column and a row, and the entry in each cell is the coefficient of the variable corresponding to that column (or is the right-hand side element) in the constraint (or objective function) corresponding to that row. It is useful to know the number of nonzero cells in the matrix because only nonzero cell entries have to be punched on cards for implementing the model on a computer, and the number of nonzero cells affects the computation time for solving the problem. The number of nonzero cells given in Table 4 assumes that prison guards are permitted to work three consecutive periods.

TABLE 4

Summary of the Mathematical Model

Number of variables

N		1
x		21
y		21
z		21
u	8 x 21	= 168
w	28 x 21	= 588
Total,	1 + (39 x 21)	= 820

Number of constraints

Total Work Force		1
SunntR73, etc. (21 work-force requirements)		21
Y73, etc.		21
X11, etc.		21
Z73, etc.		21
U5211, etc.	7 x 21	= 147
Total	1 + (11 x 21)	= 232

Number of nonzero cells in the matrix

Objective function	1 + (2 x 21)	= 43
Total work force	1 + 21	= 22
SunntR73, etc.	8 x 21	= 168
Y73, etc.	9 x 21	= 189
X11, etc.	9 x 21	= 189
Z73, etc.	29 x 21	= 609
U5211, etc.	35 x 21	= 735
Total	2 + (93 x 21)	= 1955

Exercise 3.5 Summary of the Mathematical Model When Prison Guards Must Take At Least Two Rest Periods Between Work Periods:

Reconstruct Table 4 for this case. How many cells are in the matrix now? How many have nonzero entries?

3.6 Solution Procedure

The solution procedure is as follows. The linear programming problem is solved, ignoring the integer character of the variables. This yields a global optimal solution, but will in general produce noninteger values for the variables. If the resulting total work force N is noninteger, it is rounded up to the next largest integer value. Whatever (fractional) number that was added to N to accomplish this rounding is then added to x_{11}; this ensures that the x's will still sum to the new (integer) value of N (the choice of x_{11} is arbitrary). Each of these twenty-one x's which is noninteger is then rounded to either the next smallest or next largest integer in such a way that the integer sum N of all twenty-one x's is preserved. A FORTRAN program to accomplish this rounding is given in Table 5. The linear programming problem is then resolved with the x's fixed at these integer values. Since the work-force requirements (the R_{ij}'s) are integers, this results in integer-valued overtime assignments; specifically, the sum $y_{ij} + z_{ij}$ now necessarily must be a nonnegative integer for each of the 21 periods. Furthermore, minimization of total labor cost requires that no double-time assignment be made for any period in which time-and-a-half assignments are still possible. Since the time-and-a-half variables (the u's) satisfy

$$\sum_{i,j} u_{ij;pq} \leq x_{pq}$$

where the x's are integers, exhaustion of all possible time-and-a-half assignments implies that the u's will be integers; hence each

$$y_{ij} = \sum_{p,q} u_{ij;pq}$$

will be an integer. Then each z_{ij} will be an integer.

TABLE 5

FORTRAN Program for Rounding the x's

```
C
C      ROUND THE OPTIMAL X'S TO INTEGER VALUES.
C
       CHARACTER*19 DAYS(7)/'   MONDAY-FRIDAY    ','  TUESDAY-SATURDAY ',
      1'WEDNESDAY-SUNDAY   ',' THURSDAY-MONDAY    ','   FRIDAY-TUESDAY  ',
      2' SATURDAY-WEDNESDAY',' SUNDAY-THURSDAY '/
       CHARACTER*9 SHIFTS(3)/' MORNING ','AFTERNOON',' NIGHT   '/
       DIMENSION XX(21)
       SUMX=0.0
       SUMC=0.0
C      SUMC IS THE ALGEBRAIC TOTAL OF 'CREDITS' FROM ROUNDING THE X'S DOWN (+) OR
C      UP (-).  WHEN ALL 21 OF THE X'S HAVE BEEN ROUNDED, SUMC WILL BE ZERO AND
C      THE TOTAL OF THE ROUNDED INTEGER X'S (SUMX) WILL EQUAL THE TOTAL OF THE
C      ORIGINAL OPTIMAL X'S (I.E., THE TOTAL WORKFORCE).  IT IS NOT NECESSARY TO
C      KNOW THE TOTAL WORKFORCE TO DO THE ROUNDING.
       PUNCH 5
     5 FORMAT (6HBOUNDS)
       READ 15, (XX(L),L=1,21)
    15 FORMAT (5X,F10.5)
       DO 60 I=1,3
       DO 60 J=1,7
C      TAKE THE X'S IN ORDER BY SHIFT.
       K=I+3*(J-1)
       X=XX(K)
       IX=X
       FIX=IX
C      IX AND FIX ARE THE INTEGER PART OF X IN INTEGER AND FLOATING POINT MODES.
       FX=X-FIX
C      FX IS THE FRACTIONAL PART OF X.
       IF (FX .EQ. 0.0) GO TO 50
       IF (FX .LT. 0.5 .AND. SUMC .LT. (1.0-FX)) GO TO 10
       IF (FX .LT. 0.5 .AND. SUMC .GE. (1.0-FX)) GO TO 20
       IF (FX .GE. 0.5 .AND. (-SUMC) .LT. (1.0-FX)) GO TO 20
       IF (FX .GE. 0.5 .AND. (-SUMC) .GE. (1.0-FX)) GO TO 10
    10 X=FIX
C      X IS ROUNDED DOWN.
       SUMC=SUMC+FX
       GO TO 50
    20 X=FIX+1.0
C      X IS ROUNDED UP.
       SUMC=SUMC-(1.0-FX)
    50 SUMX=SUMX+X
       XX(K)=X
    60 CONTINUE
       K=0
       DO 100 I=1,7
       DO 100 J=1,3
       K=K+1
       PRINT 55, I,J,XX(K),DAYS(I),SHIFTS(J)
    55 FORMAT (2H0X,I1,I1,3H = ,F10.5,15H OFFICERS WORK ,A19,4H ON ,A9,
      124H SHIFT AT STANDARD RATE.)
       PUNCH 65, I,J,XX(K)
    65 FORMAT (7H FX RND,7X,1HX,I1,I1,7X,F12.8)
   100 CONTINUE
       PRINT 75, SUMX
    75 FORMAT (//1H0,F15.5,19H = TOTAL WORKFORCE.)
       RETURN
       END
```

But

$$z_{ij} = \sum_{m,n} \sum_{p,q} w_{ij(mn;pq)}$$

where

$$\sum_{i,j} w_{ij(mn;pq)} \le u_{mn;pq}.$$

Since the u's and z's are integers, this implies that the w's will be integers. Thus this second linear programming problem (using the rounded integer-valued x's) will possess a completely integer-valued optimal solution, and this solution is used as the solution for the model.

In solving an integer linear programming problem, it is known that merely rounding-off noninteger variables in the solution to the corresponding linear programming problem can result in an infeasible solution for the integer problem, or a feasible solution which is far from the optimal integer-valued solution ("far" as measured by the value of the objective function). For a simple graphical illustration of this, see pages 687-698 in the book by Hillier and Lieberman (see References). However, we are not taking these risks in our procedure. By rounding the noninteger x's and N so that the constraint

$$\sum_{k,j} x_{kj} = N$$

is preserved, we are sure that any resulting optimal solution to the second linear programming problem is both feasible for the original problem, and also optimal for the original problem <u>given these integer x-values</u>.

Also, from the (noninteger-valued) solution to the first problem, we have the global minimum total labor cost for comparison with the minimum total labor cost from the second (integer-valued) solution. In all test cases that have been run, the increase in total labor cost between the global optimal solution and the integer-valued solution has been nominal. For example, the increase in total labor cost between the global optimal solution and the integer-valued solution for the data in Section 4.2 was only 2.25 times the cost of a standard-rate eight-hour period, i.e., 2.25 x \$24 = \$54, if prison guards were paid \$3 per hour as standard rate.

In other words, since we do not know that the rounded integer-valued x's used in the second linear programming problem are optimal <u>integer</u> values for the original problem, we cannot be sure that we have the optimal integer-valued solution to the original problem. But we do know that we have a very good integer-valued solution, since it is almost as good (measured by the objective function) as the global optimal solution (the optimal solution to the first problem). In solving real-world problems, this is often the best that can be done.

Exercise 3.6 Rounding the x's

(For students who know FORTRAN): Given the following twenty-one x-values, whose sum is N = 137, round them to all integer values by following the FORTRAN program in Table 5. Verify that the sum of the rounded integer x's is still 137.

$x_{11} = 17.2$	$x_{12} = 9.4$	$x_{13} = 3.0$
$x_{21} = 8.2$	$x_{22} = 6.4$	$x_{23} = 2.8$
$x_{31} = 10.2$	$x_{32} = 8.4$	$x_{33} = 0.0$
$x_{41} = 3.2$	$x_{42} = 6.4$	$x_{43} = 3.8$
$x_{51} = 6.2$	$x_{52} = 9.4$	$x_{53} = 6.4$
$x_{61} = 9.2$	$x_{62} = 7.4$	$x_{63} = 3.0$
$x_{71} = 4.2$	$x_{72} = 8.4$	$x_{73} = 3.8$

---

4. COMPARISON OF RESULTS FROM THE MODEL WITH PAST DATA FROM TWO PRISONS

4.1 Prison H

Data provided by prison H for the week ending September 23, 1973 are shown in Table 1. For each eight-hour work period, eight numbers are shown. Recall from Section 1.2 that the first, second, and third numbers not in parentheses are the numbers of prison guards reported as having been assigned to that period at standard rate, one-and-a-half-rate overtime, and double-rate overtime, respectively. The fourth number not in parentheses for each eight-hour period is the sum of the first three, and was used in the model as the work-force requirement (R_{ij}) for that period (e.g., $R_{31} = 49$).

For this test of the model, it was decided to use the total work force N fixed at a value representative of the total work force actually available at Prison H during that week, i.e., to modify the basic model as discussed in Section 2.3. This modification merely requires that the variable N in the total work-force constraint ($\Sigma\Sigma x_{kj} = N$) be replaced by the desired numerical value, and the objective function be modified to include overtime cost only, i.e., $Z = \Sigma\Sigma(1.5y_{ij} + 2z_{ij})$, since the cost of standard-rate work is fixed when N is fixed. Unfortunately, the value of N could not be provided by the prison management. Under the assumption that the standard-rate work assignments reported in the data represent each guard working a forty-hour work schedule as described in assumption (3) of Section 2.1, it is possible to solve for the number of guards assigned to each of the twenty-one forty-hour work schedules, i.e., to solve for the twenty-one x's.

For example, using the data from Morning Shift in Table 1 and the standard-rate assignments (the x's) indicated in Table 3, we can write

$$
\begin{aligned}
&\text{Monday:} && x_{11} && && && + x_{41} && + x_{51} && + x_{61} && + x_{71} && = 40\\
&\text{Tuesday:} && x_{11} && + x_{21} && && && + x_{51} && + x_{61} && + x_{71} && = 44\\
&\text{Wednesday:} && x_{11} && + x_{21} && + x_{31} && && && + x_{61} && + x_{71} && = 44\\
&\text{Thursday:} && x_{11} && + x_{21} && + x_{31} && + x_{41} && && && x_{71} && = 42\\
&\text{Friday:} && x_{11} && + x_{21} && + x_{31} && + x_{41} && + x_{51} && && && = 44\\
&\text{Saturday:} && && x_{21} && + x_{31} && + x_{41} && + x_{51} && + x_{61} && && = 36\\
&\text{Sunday:} && && && x_{31} && + x_{41} && + x_{51} && + x_{61} && + x_{71} && = 33
\end{aligned}
$$

These seven equations can then be solved for the seven x's. This same procedure can be used for Afternoon shift and Night shift. Then the sum of these twenty-one x's should give the total work force N (under assumption (3) of Section 2.1).

Note that the coefficient matrix of the above equations is

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1\\
1 & 1 & 0 & 0 & 1 & 1 & 1\\
1 & 1 & 1 & 0 & 0 & 1 & 1\\
1 & 1 & 1 & 1 & 0 & 0 & 1\\
1 & 1 & 1 & 1 & 1 & 0 & 0\\
0 & 1 & 1 & 1 & 1 & 1 & 0\\
0 & 0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

whose inverse is

$$
\frac{1}{5}
\begin{pmatrix}
3 & -2 & 3 & -2 & 3 & -2 & -2\\
-2 & 3 & -2 & 3 & -2 & 3 & -2\\
-2 & -2 & 3 & -2 & 3 & -2 & 3\\
3 & -2 & -2 & 3 & -2 & 3 & -2\\
-2 & 3 & -2 & -2 & 3 & -2 & 3\\
3 & -2 & 3 & -2 & -2 & 3 & -2\\
-2 & 3 & -2 & 3 & -2 & -2 & 3
\end{pmatrix}
$$

and these two matrices will also apply for the Afternoon shift and Night shift data.

However, when this was done for the Prison H data in Table 1, fourteen of those twenty-one numbers (the x's for Morning and Night shifts) were fractional (e.g., x_{11} = 14 4/5), indicating that the data are not consistent with assumption (3); indeed, it is now known that the two state prisons discussed here (Prisons H and G) were not then scheduling guards according to assumption (3). However, even if assumption (3) had been in effect, absenteeism could still result in some such inconsistencies. It was then necessary to estimate the total work force by some method. This was done by summing the twenty-one numbers (including their fractional parts) as found from the above procedure, and then rounding this noninteger sum to the next largest integer, which gave an estimated total work force of N = 137.

Since each prison guard is assumed to work one eight-hour period per day for five consecutive days at standard rate, this same estimate can be obtained by dividing the sum of the twenty-one standard-rate work assignments (as reported by Prison H) by five, and then rounding this noninteger quotient to the next largest integer. For example, add the previously-given equations for the Morning shift x's, repeat for the other two shifts, and add all three of these results; on the left you have 5 times the sum of the 21 x's and on the right you have the numerical sum of the twenty-one standard-rate work assignments from the data. However, this method does not explicitly reveal the degree to which the data are inconsistent with assumption (3). To illustrate this method using the data from Table 1, we have (40 + 44 + 44 + 42 + ... + 17 + 16 + 16 + 17)/5 = 136.2 so take N = 137, as above.

Using this total work force and the work-force requirements indicated in Table 1, the model was applied to find a set of "optimal" work-force assignments that would satisfy these requirements at "minimum" overtime cost (which is equivalent to minimizing total labor cost, because here the total work force was taken as fixed at 137).

For each eight-hour work period in Table 1, the model results for standard-rate, one-and-a-half-rate overtime, and double-rate overtime assignments are shown as the first, second, and third numbers in parentheses, respectively. The fourth number in parentheses is the total of the first three, and must equal or exceed the fourth number not in parentheses (R_{ij}), for the work-force requirement to be met during that eight-hour period. Of course, the model guarantees that this will always occur.

The week ending September 23, 1973 was chosen for study at Prison H because the data for this week indicated more double-rate overtime than any other week at Prison H during the period for which data were reported (from the week ending July 1, 1973 to the week ending December 2, 1973). Assuming that prison guards are paid $3 per hour as standard rate, the total overtime cost indicated by the data in Table 1 is $564, as found in Section 1.2. The total overtime cost that would have resulted from the model solution (based on a total work force of 137) is

$$(1.5)(1 + 5 + 1 + 3)(\$24) = \$360$$

for a savings of $204 during this week at Prison H. If prison guards were paid more than $3 per hour as standard rate, the savings would be proportionately larger. The computer cost to obtain this solution was about $23, but with a more sophisticated computer implementation of the model (which was used with the Prison G data described in Section 4.2), this computing cost probably would have been less than $11.

<u>Exercise 4.1 Solving for the "Optimal" Standard-Rate Assignments</u>

(a) Using the given inverse matrix, verify that the data in Table 1 yield $x_{11} = 14\ 4/5$.

(b) Exercise 3.6 gave the noninteger global optimal x's resulting from N = 137 and the R_{ij}'s indicated in Table 1. In Exercise 3.6 you rounded these x's to all integer values. Now verify that these integer x's result in the "optimal" standard-rate work assignments shown in Table 1 (the first number in parentheses in each eight-hour period).

4.2 Prison G

Data provided by Prison G for the week ending September 30, 1973 are shown in Table 2. The arrangement of the data is similar to that in Table 1. An attempt to determine the total work force N from the standard-rate work assignments shown in Table 2 using the method described in Section 4.1 showed these data to be wildly inconsistent with assumption (3) of Section 2.1. Not only were seven of the twenty-one numbers (the x's, representing forty-hour work-schedule assignments) fractional, but four of them were negative. So, for this set of data it was decided to have the model determine the "optimal" size of the work force N as well as the "optimal" assignments to satisfy the work-force requirements in Table 2, at "minimum" total labor cost. This is the basic model with no modifications.

The "optimal-sized" work force N for "minimum" total labor cost was found to be 349; the "optimal" assignments are shown in parentheses in Table 2 (as was done in Table 1).

In order to compare the apparent actual total labor cost for this week at Prison G with the cost resulting from the model solution, it was again necessary to estimate the actual total work force by some method. As in Section 4.1, the method used was to algebraically sum the twenty-one numbers (x's) representing the forty-hour work-schedule assignments (including their fractional parts and taking account of the negative values), and then round this sum to the next largest integer, which gave an estimated total work force of 238.

Again assuming that prison guards are paid \$3 per hour as standard rate, the total labor cost indicated by the data reported by Prison G for this week is

$$(6)(238)(\$24) + (1.5)(542)(\$24) + (2)(92)(\$24) = \$58,200$$

(based on the estimated total work force of 238). The total labor cost that would have resulted from the model solution is

$$(6)(349)(\$24) + (1.5)(112)(\$24) = \$54,288$$

for a savings of \$3,912 during this week at Prison G. The computing cost to obtain this solution was less than \$10. Thus, a substantial savings in total labor cost could be obtained by increasing the work force and reducing the amount of overtime work in an "optimal" manner.

4.3 Comments on the Model Results

For the two tests of the model described here, the model was implemented on the Pennsylvania State University's IBM 370/168 running under OS and using the Mathematical Programming System/360 Version 2, Linear Programming. Computing expenses were nominal, as reported in Sections 4.1 and 4.2.

Overstaffed periods, as discussed in Section 2.5, occur in the two examples just presented as follows:

(A) At Prison H - Thursday Afternoon.
(B) At Prison G - Monday Morning, Friday Morning, Friday Afternoon, and Sunday Night.

It should be noted that some of these overstaffed periods result from the rounding procedure used to obtain an integer-valued solution, while others (notably Friday Afternoon and Sunday Night at Prison G) occur because of the block effect in the standard-rate work assignments as discussed in Section 2.5. That is, the overstaffing for Friday Afternoon and Sunday Night at Prison G occurs in the noninteger-valued global optimal solution as well as in the "optimal" integer-valued solution given in Table 2.

The model solutions given in Tables 1 and 2 indicate no double-rate overtime assignments. However, this need not always occur. For example, if the case discussed in Section 4.2 is repeated, but with a Tuesday Afternoon work-force requirement of 46 instead of 146, then the "optimal" total work force is N = 304, and double-rate overtime assignments will occur for Wednesday Night and Thursday Morning.

When evaluating the cost savings indicated in these two examples, it must be remembered that the data provided by the two prisons most likely do not correspond to the estimated total work forces used here for comparison purposes (137 at Prison H and 238 at Prison G), definitely do not correspond to assumption (3) of Section 2.1, and may not correspond to assumption (5) either. Nevertheless, in each example the model has provided a clear reduction in the number and cost of overtime assignments while at least meeting the same total work-force requirements for each eight-hour period of the week.

Exercise 4.3 Partial Overtime Periods

In some situations, such as manufacturing plants, it is customary to have employees work overtime for partial periods (i.e., less than eight hours) until the unfinished work is completed. For the present model, assumption (4) of Section 2.1 stated that each overtime assignment is to be for one (full) eight-hour period. For this model, do you think there would be any advantage, in terms of total labor cost, in permitting prison guards to work partial overtime periods? Explain.

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5. CONCLUDING REMARKS

The results discussed in Chapter 4 indicated that considerable savings could be achieved through the use of this model. In some cases large savings would result from increasing the size of the total work force from its present value to a larger "optimal" value, since the resulting reduction in overtime cost would more than offset the increased costs of standard-rate work and fringe benefits for the additional employees; administrators who are laughing under externally-imposed hiring "freezes" may be familiar with this situation.

The model, of course, is applicable to any work-scheduling situation satisfying the assumptions or their modifications described in Section 2. For example, work-scheduling situations involving medical personnel, police forces, fire-fighting crews, and other emergency personnel may exhibit characteristics similar to the present case; such as the need for round-the-clock staffing and the meeting of pre-specified minimum work-force requirements which repeat in a cyclic pattern.

This model, along with some others, is briefly discussed in the article "Applications of Operations Research Methods to Correctional Problems" by Sitansu S. Mittra (Criminal Justice and Behavior, Vol. 2, No. 2, June 1975, pp. 169-179).

The author presented a paper discussing this model at the 1977 North Central Section/MAA Summer Seminar on Model Building, at Bemidji State University, Bemidji, Minnesota, June 20-24, 1977. That paper, and all others presented there, appear in the transactions of the seminar.

REFERENCES

[1] Linear Programming, by G. Hadley (Addison-Wesley, Reading, Massachusetts; 1962).

[2] "Manpower Allocations to Meet Cyclic Requirements" by J. A. Parsons. Journal of Systems Management, Vol. 27, No. 6, Issue No. 182, June 1976, pp. 26-27.

[3] Operations Research, second edition, by Frederick S. Hillier and Gerald J. Lieberman (Holden-Day, San Fransisco; 1974).

[4] Operations Research, An Introduction, by Hamdy A. Taha (Macmillan, New York; first edition 1971; second edition 1976).

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FINAL EXAM

Consider the following "small-scale" version of the prison guard scheduling problem (and compare the following ten assumptions with those in Section 2.1):

(1) Each twenty-four hour day is divided into two twelve-hour shifts: Morning and Evening (see Table 6).

(2) Work-force requirements are known for each twelve-hour period of each day in a repeating four-day cycle.

Since we are concerned here with a four-day "week" instead of the usual seven-day week, we will not use the usual names for the days of the week. Instead of Monday, Tuesday, etc., we will name the days in our four-day "week" Oneday, Twosday, Threesday, Foursday, and abbreviate these names as O, T, Th, F, respectively (see Table 6).

These known work-force requirements do not change from "week" to "week.". Also, the total number of prison guards available for work is constant.

(3) As a standard "weekly" tour of duty, each prison guard is assigned to a twenty-four-hour standard-rate work schedule consisting of one twelve-hour period per day for two consecutive days, staying on the same shift for both days; he/she then has two consecutive days off before beginning this same pattern again. These standard-rate work schedules repeat "week" after "week."

(4) Work-force requirements not met by these standard-rate work-schedules are filled by overtime assignments; each overtime assignment is for one twelve-hour period.

(5) A prison guard is not eligible for overtime work on any of the three periods which occur during his/her standard twenty-four-hour work "week."

(6) A prison guard is eligible for overtime work on any of the five periods which occur during his/her two days off.

(7) However, no prison guard may work more than two overtime periods during his/her two days off.

(8) A prison guard may work more two or even three consecutive twelve-hour periods, depending on what

TABLE 6

A "Small-Scale" Scheduling Problem

Day	Shift	
	Morning (a.m.) 12 midnight - 12 noon	Evening (p.m.) 12 noon - 12 midnight
Oneday		
Twosday		
Threesday		
Foursday		

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combination of standard and overtime shifts he/she is assigned.

(9) All prison guards are assumed to be paid at the same (unspecified) standard rate for a twelve-hour period.

(10) If a prison guard does overtime work during his/her two days off, then for the first overtime periods during the two days off he/she is paid one-and-a-half (1.5) times his/her standard rate of pay. If he/she works a second overtime period during the same two days off, then for this second overtime period he/she is paid two (2) times his/her standard rate of pay.

The four possible two-day work schedules are indicated below, with a corresponding value for an index k (compare with Section 3.2):

Work Schedule				k
O	T			1
	T	Th		2
		Th	F	3
O			F	4

A notation similar to that defined in Section 3.2 should be used for the following exercises.

1. Complete Table 6 for the present "small-scale" prison guard scheduling problem, in a manner similar to Table 3.

2. Assuming that fringe benefits are equal to 25% of a prison guard's standard-rate pay, and are a cost in addition to his/her standard-rate pay, write out the objective function.

3. Write out in full the following constraints for this model:

$$\Sigma\Sigma x_{kj} = N \qquad \Sigma u_{??;??} \leq x_{11}$$

$$\sum_k x_{kj} + y_{??} + z_{??} \geq R_{11} \qquad z_{11} = \Sigma\Sigma w_{??(??;??)}$$

$$y_{11} = \Sigma u_{??;??} \qquad \Sigma w_{??(??;??)} \leq u_{11;22}$$

4. In the constraints of Exercise 3, underline the entries (as in Section 3.4) to satisfy the following:

If prison guards are permitted to work three consecutive periods, retain all entries. If guards are permitted to work two, but not three, consecutive periods, then delete all triply underlined entries. If guards are required to have at least one rest period between work periods, then delete all triply and doubly-underlined entries. Finally, if guards are required to have at least two rest periods between any standard-rate work period and any overtime period, delete all underlined entries (how does this last requirement affect the model?).

5. Construct a tabular summary of the model similar to Table 4, assuming that guards are permitted to work three consecutive periods.